UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

BEng (HONS) MECHANICAL, ELECTRICAL & ELECTRONIC ENGINEERING

SEMESTER ONE EXAMINATION 2024/25

ENGINEERING MODELLING AND ANALYSIS

MODULE NO: AME5014

Date: Monday 13th January 2025 **Time: 10:00am – 12:00pm**

INSTRUCTIONS TO CANDIDATES:

There are EIGHT questions.

Answer ANY FIVE questions.

All questions carry equal marks.

Marks for parts of questions are shown in brackets.

Electronic calculators may be used if data and program storage memory is cleared prior to the examination.

CANDIDATES REQUIRE:

Formula Sheets (attached following questions).

School of Engineering
BEng Mechanical, Electrical and Electronic Engineering
Semester One Examination 2024/2025
Engineering Modelling and Analysis
Module No. AME5014

Q1: Differentiation-Integration

(a) In an RLC circuit, the charge $Q(t)$ in C on the capacitor varies with time according to the equation:

$$Q(t) = 4e^{-3t}sin(10t)$$

where where t is the time in s. The current $i(t)$ in A in the circuit is the rate of change of charge, i.e.,

$$i(t) = \frac{dQ}{dt}$$

Differentiate the given equation $Q(t) = 4e^{-3t}sin(10t)$ to find $i(t)$

(10 Marks)

(b) The torque τ in Nm required to rotate a circular disk varies with the angular displacement θ and is given by:

$$\tau(\theta) = 5\theta^2 + 3\theta$$

where θ is the angular displacement in rad. Calculate the work done, W, in J in rotating the disk from $\theta = 0\,rad$ to $\theta = 2\,rad$

Hint:

- The work done by the torque is given by:

$$W = \int_{\theta_1}^{\theta_2} \tau(\theta)d\theta$$

(10 Marks)

Total 20 Marks

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Q2: Second Order Differential Equation

Solve ONE of the TWO parts below:

Part 1:

(a) A mass-spring-damper system consists of a mass $m = 2\,kg$, a spring with stiffness $k = 20\,N/m$, and a damper with a damping coefficient $c = 5\,Ns/m$. The displacement $x(t)$ of the mass is described by the following second-order differential equation:

$$mx''(t) - cx'(t) - kx(t) = 0$$

where $x(t)$ is the displacement from the equilibrium position in m, and t is time in s.

Given the initial conditions $x(0) = 0.1\,m$ and $x'(0) = 0\,m/s$, solve the equation analytically to determine the displacement $x(t)$, and describe the nature of the system's response (e.g., underdamped, overdamped, or critically damped).

(14 Marks)

(b) Create a table listing the displacement $x(t)$ at $t = 1\,s$, $2\,s$ and $3\,s$, based on your solution from part 1(a).

(6 Marks)

Total 20 Marks

Question 2 continues in the next page…

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Q2 continued

Part 2:

(a) In a DC motor circuit, the electrical dynamics of the armature are modeled by the following second-order differential equation:

$$L\frac{d^2i(t)}{dt^2} - R\frac{di(t)}{dt} - ki(t) = 0$$

where:

- $i(t)$ is the current through the motor's armature as a function of time,
- $L = 0.2\,H$ is the inductance of the motor,
- $R = 5\,\Omega$ is the resistance of the motor's armature, and
- $K = 50$ is a constant related to the back electromotive force (EMF).

Given the initial conditions:

$$i(0) = 3\,A \text{ and } \frac{di(t)}{dt} = 0\,A/s$$

Solve the equation analytically to find the current $i(t)$ as a function of time and describe the nature of the current response in the motor's armature (underdamped, overdamped, or critically damped).

(14 Marks)

(b) Create a table listing the current $i(t)$ for $t = 0.5\,s$, $1\,s$, and $2\,s$, based on your solution from part 2(a).

(6 Marks)

Total 20 Marks

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Q3: First Order Differential Equation

Solve <u>ONE</u> of the <u>TWO</u> parts below:

<u>Part 1:</u>

(a) A cooling object is placed in a room with a constant ambient temperature of $T_{room} = 20^\circ\ C$. According to Newton's Law of Cooling, the rate at which the temperature $T(t)$ of the object changes over time is proportional to the difference between the object's temperature and the ambient temperature. The relationship is given by:

$$\frac{dT}{dt} = -k(T(t) - T_{room})$$

where:

- k is the cooling constant, with a value of $0.1\, min^{-1}$.
- $T(t)$ is the temperature of the object at time t (in min).

Given that the initial temperature of the object is $T(0) = 100^\circ C$, solve for $T(t)$ as a function of time t.

(12 Marks)

(b) Determine the temperature of the object after $t = 10\, min$ and $t = 20\, min$.

(8 Marks)

Total 20 Marks

Question 3 continues in the next page…

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Q3 continued

Part 2:

(a) A capacitor with capacitance $C = 0.01\,F$ is connected in series with a resistor $R = 200\,\Omega$ in an RC circuit. Initially, the capacitor has a charge such that the voltage across it is $V_C(0) = 100\,V$ at $t = 0\,s$. The voltage across the capacitor is allowed to discharge through the resistor after being disconnected from the power source.

The voltage across the capacitor $V_C(t)$ is governed by the following differential equation:

$$RC\frac{dV_C}{dt} + V_C(t) = 0$$

Substitute the values $R = 200\,\Omega$ and $C = 0.01\,F$ into the equation and find the general solution for $V_C(t)$.

(12 Marks)

(b) Given that at $t = 0\,s$, the capacitor voltage $V_C(0)$ is $100\,V$, find the fully defined solution for $V_C\ (t)$ and determine the time constant, $\tau = RC$ in s of the circuit.

(8 Marks)

Total 20 Marks

Q4: Laplace Transforms

Solve <u>ONE</u> of the <u>TWO</u> parts below:

<u>Part 1:</u>

(a) A damped mass-spring system is modeled such that the velocity $v(t)$ of a mass $m = 5\,kg$ is influenced by a linear damper with a damping constant $c = 0.1\,Ns/m$. The governing equation for the velocity $v(t)$ is:

$$\frac{dv(t)}{dt} = \frac{c}{m}v(t)$$

Given that the mass initially $(t = 0\ s)$ has a velocity of $10\,m/s$, use the method of **Laplace transforms** to derive an expression for $v(t)$.

(12Marks)

(b) Estimate the time t (in s) it takes for the velocity of the mass to decrease to $2\,m/s$.

(8 Marks)

Total 20 Marks

Question 4 continues in the next page…

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Q4 continued

Part 2:

(a) Acharging inductor in an RL circuit is modeled by the following differential equation:

$$5\frac{dI}{dt} + 15I = 90$$

where I (in A) is the instantaneous current through the inductor at time t (in s).

Given: initial condition, $I(0) = 0\ A$ when $t = 0\ s$.

Use the method of **Laplace transforms** to derive an expression for $I(t)$.

(12 Marks)

(b) Estimate the time t (in s) taken for the I to reach $4\ A$.

(8 Marks)

Total 20 Marks

Q5: Fourier transform

A electro-mechanical vibration system generates a square pulse force $F(t)$ applied to a mass for a short duration. The force is modeled as:

$$F(t) = 10, \qquad for\ -2 \le t \le 2$$
$$f(t) = 0, \qquad \text{otherwise}$$

(a) Sketch the waveform of the force signal $F(t)$ and describe its key characteristics.

(6 Marks)

(b) Calculate the Fourier transform $F(\omega)$ of the force signal and comment on the frequency components of the force.

(14 Marks)

Total 20 Marks

Q6: Matrices

Solve <u>ONE</u> of the <u>TWO</u> parts below:

<u>Part 1:</u>

A feedback control system for an amplifier circuit is represented by the following differential equation:

$$\frac{d\vec{x}}{dt} = D\vec{x}$$

where,

$$\vec{x}(t) = e^{\lambda t}\vec{\omega}$$

$$D = \begin{pmatrix} -3 & 2 \\ -1 & -4 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

- t is time in s.
- λ is an eigenvalue.
- $\vec{v}$ is an eigenvector.
- V_1 and V_2 represent the voltages in V at different points in the amplifier circuit.

a) Find the eigenvalues of matrix D.

(8 Marks)

b) Find the eigenvectors of matrix D.

(12 Marks)

Total 20 Marks

Question 6 continues in the next page…

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Q6 continued

<u>Part 2:</u>

The stability of a robotic arm with two joints is modelled by the following equation:

$$\frac{d\vec{x}}{dt} = E\vec{x}$$

where:

$$E = \begin{pmatrix} -10 & 3 \\ 4 & -6 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

- θ_1 and θ_2 represent the angular displacements of the joints in rad.
- t is time in s.
- λ is the eigenvalue used to assess the stability of the system.

(a) Find the eigenvalues of matrix E .

(10 Marks)

(b) Find the eigenvectors of matrix E.

(10 Marks)
Total 20 marks

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Q7: Simpson's rule

Solve ONE of the TWO parts below:

Part 1:

The speed v of a piston in an engine varies with time t according to the table below:

Time - t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
Speed- v (m/s)	2.0	3.5	4.8	5.4	6.0	7.2	8.0

(a) Sketch the graph of speed v versus time t from the data given in the table and annotate the graph appropriately.

(6 Marks)

(b) Find an approximate value for the displacement in m of the piston $\int_0^3 v\,dt$ using Simpson's rule.

(14 Marks)

Total 20 Marks

Part 2:

The mean current, $\bar{\imath}$, flowing through a circuit is given by:

$$\bar{\imath} = \frac{1}{0.4}\int_0^{0.5} v\,dt$$

where, v is values of voltages measured at intervals of 0.1s as shown in the table below.

Time - t (s)	0	0.1	0.2	0.3	0.4	0.5
Potential- v (V)	2.5	2.4	3.35	2.3	2.25	2.2

(a) Sketch the graph of the potential, v, versus time, t, from the data given in the table and annotate the graph appropriately.

(6 Marks)

(b) Find an approximate value for the mean current, $\bar{\imath}$, in A using Simpson's rule.

(**14 marks**)

Total 20 Marks

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Q8: Partial derivative and double integrals

(a) A heat transfer problem, the temperature T (in $°C$) at a point in a metal rod depends on both position x (in m) and time t (in s). The temperature distribution is given by:

$$T(x,t) = e^{-0.5t} * \sin(x)$$

Evaluate Z, so that:

$$Z = \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x \partial T}$$

If $x = \pi/4$ and $y = 2\ s$.

(**10 Marks**)

(b) A function $f(x,y)$ represents the distribution of charge density (in C/m^2) on a square plate. The function is defined as:

$$f(x,y) = \int_{x=0}^{x=5} \int_{y=0}^{y=7} (4\,x^2 - 3\,y^3 + 2)\,dy\,dx$$

(**10 Marks**)

Total 20 Marks

<u>END OF QUESTIONS</u>

FORMULA SHEET FOLLOWS ON NEXT PAGES

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FORMULA SHEET

Partial Fractions

$$\frac{F(x)}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\frac{F(x)}{(x+a)(x+b)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2}$$

$$\frac{F(x)}{(x^2+a)} = \frac{Ax+B}{(x^2+a)}$$

Small Changes

z = f(u, v, w)

$$\delta z \simeq \frac{\partial z}{\partial u}.\delta u + \frac{\partial z}{\partial v}.\delta v + \frac{\partial z}{\partial w}.\delta w$$

Total Differential

z = f(u, v, w)

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

Rate of Change

z = f(u, v, w)

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}.\frac{du}{dt} + \frac{\partial z}{\partial v}.\frac{dv}{dt} + \frac{\partial z}{\partial w}.\frac{dw}{dt}$$

Eigenvalues

$$|A - \lambda \mathrm{I}| = 0$$

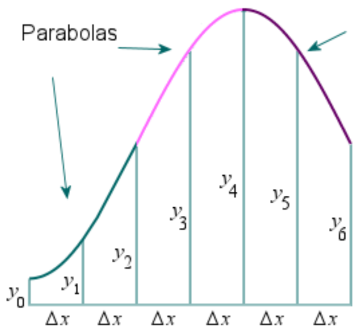
Eigenvectors

$$(A - \lambda_r I)x_r = 0$$

Integration

$$\int u.\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}dx$$

Simpson's rule
To calculate the area under the curve which is the integral of the function **Simpson's Rule** is used as shown in the figure below:



The area into *n* equal segments of width *Δx*. Note that in Simpson's Rule, *n* must be EVEN. The approximate area is given by the following rule:

$$Area = \int_a^b f(x)dx = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 \ldots + 4y_{n-1} + y_n)$$

Where $\Delta x = \frac{b-a}{n}$

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Differential equation

Homogeneous form:

$$a\ddot{y} + b\dot{y} + cy = 0$$

Characteristic equation:

$$a\lambda^2 + b\lambda + c = 0$$

Quadratic solutions :

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i. If $b^2 - 4ac > 0$, λ_1 and λ_2 are distinct real numbers then the general solution of the differential equation is:

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

A and B are constants.

ii. If $b^2 - 4ac = 0$, $\lambda_1 = \lambda_2 = \lambda$ then the general solution of the differential equation is:

$$y(t) = e^{\lambda t}(A + Bx)$$

A and B are constants.

iii. If $b^2 - 4ac < 0$, λ_1 and λ_2 are complex numbers then the general solution of the differential equation is:

$$y(t) = e^{\alpha t}[Acos(\beta t) + Bsin(\beta t)]$$

$$\alpha = \frac{-b}{2a} \quad and \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

A and B are constants.

Inverse of 2x2 matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of A can be found using the formula:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Modelling growth and decay of engineering problem

$C(t) = C_0 e^{kt}$

$k > 0$ gives exponential growth

$k < 0$ gives exponential decay

First order system

$$y(t) = k(1 - e^{-\frac{t}{\tau}})$$

Transfer function: $\frac{k}{\tau s + 1}$

Derivatives table:

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, any constant	0
x	1
x^2	$2x$
x^3	$3x^2$
x^n, any constant n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\text{cosec}\, x = \frac{1}{\sin x}$	$-\text{cosec}\, x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$

Integral table:

$f(x)$	$\int f(x)\,\mathrm{d}x$
k, any constant	$kx + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$\mathrm{e}^x + c$
e^{kx}	$\frac{1}{k}\mathrm{e}^{kx} + c$
$\cos x$	$\sin x + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sin x$	$-\cos x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\tan x$	$\ln(\sec x) + c$
$\sec x$	$\ln(\sec x + \tan x) + c$
$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x) +$
$\cot x$	$\ln(\sin x) + c$
$\cosh x$	$\sinh x + c$
$\sinh x$	$\cosh x + c$
$\tanh x$	$\ln\cosh x + c$
$\coth x$	$\ln\sinh x + c$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$

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<u>Laplace table:</u>

$f(t)$	$F(s)$		$f(t)$	$F(s)$
1	$\frac{1}{s}$		$u_c(t)$	$\frac{e^{-cs}}{s}$
t	$\frac{1}{s^2}$		$\delta(t)$	1
t^n	$\frac{n!}{s^{n+1}}$		$\delta(t-c)$	e^{-cs}
e^{at}	$\frac{1}{s-a}$		$f'(t)$	$sF(s)-f(0)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$		$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$\cos bt$	$\frac{s}{s^2+b^2}$		$(-t)^n f(t)$	$F^{(n)}(s)$
$\sin bt$	$\frac{b}{s^2+b^2}$		$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$		$e^{ct}f(t)$	$F(s-c)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$		$\delta(t-c)f(t)$	$e^{-cs}f(c)$

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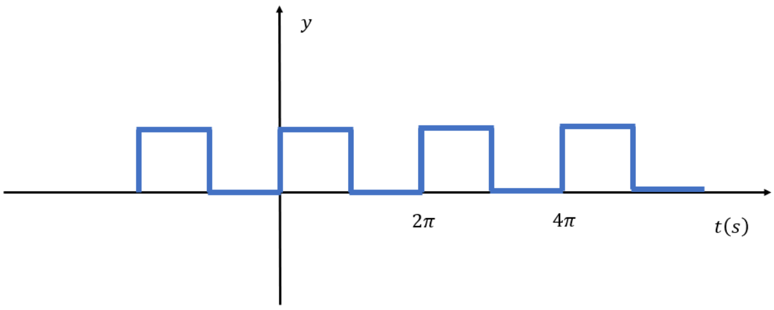
Fourier Series

The periodic square wave with Fourier Series and the coefficients of the Fourier Series



The function which represent the periodic square wave can be represented by

$$y = f(t)$$

Period of the function:

$$T = 2\pi \frac{sec}{cycle}$$

Fourier series of the function:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \cdots + a_n \cos(nt)$$
$$+ b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \cdots b_n \sin(nt)$$

Where, $n = 1,2,3,4,5, \ldots \ldots ..$

Alternatively,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

Fourier Coefficients:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(nt)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(nt)\, dt$$

Useful Equations for Fourier transform

Fourier transform equation

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

Inverse Fourier transform equation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

Euler's formula for trigonometric identities

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

Where, $j = \sqrt{-1}$

For any arbitrary function

$$\int_{a}^{b} f(t)\delta(t - t_0)\, dt = f(t_0)$$

End of the Formula Sheet

<u>END OF PAPER</u>