UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING SCIENCES

BEng (HONS) MECHANICAL ENGINEERING, BEng (HONS) ELECTRICAL & ELECTRONIC ENGINEERING

SEMESTER 1- EXAMINATION 2023/24

ENGINEERING MODELLING AND ANALYSIS

MODULE NO: AME5014

Date: Monday 8th January 2024 **Time: 14:00 – 16:00**

INSTRUCTIONS TO CANDIDATES:

There are EIGHT questions.

Answer ANY FIVE questions.

All questions carry equal marks.

Marks for parts of questions are shown in brackets.

Electronic calculators may be used if data and program storage memory is cleared prior to the examination.

CANDIDATES REQUIRE: **Formula Sheet (attached).**

School of Engineering
BEng (Hons) Mechanical Engineering
BEng (Hons) Electrical and Electronics Engineering
Semester 1 Examination 2023/2024
Engineering Modelling and Analysis
Module No. AME5014

Q1: Differentiation-Integration

(a) In an RC circuit with a capacitor of capacitance $C = 2\ F$, the voltage across the capacitor varies with time as $V(t) = 5\sin(120\pi t)\ V$. Calculate the rate of change of charge on the capacitor with respect to time, $\frac{dQ}{dt}$ in C/s.

To solve this, you would first use the equation $Q = CV$ to express the charge on the capacitor as a function of time. Then, differentiate this expression with respect to time to find $\frac{dQ}{dt}$.

[10 marks]

(b) The force F in N acting on an object varies with its position x in meters and is given by the equation $F(x) = 4x^3 - 2x$. Calculate the work done, W in J, by this force as the object moves from a position of $x_0 = 0\,m$ to $x_1 = 3\,m$.

In this question, you're asked to evaluate the work done, which requires integrating the force function over the given range of x. Therefore,

$$W = \int_{x_0}^{x_1} F(x)dx$$

[10 Marks]
Total 20 Marks

Q2: Second Order Differential Equation

Solve one of the two parts below:

Part 1:

(a) A simple pendulum with a length L of $1\ m$ and a bob of mass m of $1\ kg$, and damping coefficient c of 10 Ns/rad can be described by the following differential equation for small angles θ:

$$m\theta''(t) + c\theta'(t) + \frac{mg}{L}\theta(t) = 0$$

where $\theta(t)$ is the angular displacement from the vertical in rad, g is the acceleration due to gravity, and and its value is $9.8\ m/s^2$, and t is the time in s.

Given the initial conditions $\theta(0) = 0.18\ rad$ and $\theta'(0) = 0\ rad/s$, solve this equation analytically to find the angular displacement $\theta(t)$ and describe the nature of the pendulum's oscillatory motion.

[14 Marks]

(b) Create a table listing displacement, $\theta(t)$, found in **Q2(a) (part 1)** for $t = 1,2,3\ s$.

[6 Marks]
Total 20 Marks

Part 2:

(a) An RCL circuit in series, driven by a constant emf, with inductance, L of $1\ H$, resistance R of $50\ \Omega$ and capacitance of $0.002\ F$ can be modelled by the following equation:

$$Li''(t) + Ri'(t) + \frac{1}{C}i(t) = 0$$

For which the natural response of the circuit is $i(t)$, the current in function of the time (t). The initial conditions are $i(0) = 2\ A$ and $i'(0) = 3\ A/s$.

Solve analytically the equation above to find the current, $i(t)$, and state the nature of response of the current.

[14 Marks]

(b) Create a table listing the current, $i(t)$ found in **Q2(a) (part 2)** for $t = 1,2,3\ (sec)$.

[6 Marks]

Total 20 Marks

PLEASE TURN THE PAGE…

Q3: First Order Differential Equation

Solve one of the two parts below:

Part 1:

(a) According to Torricelli's Law, the velocity v of a fluid exiting a hole at the bottom of a tank filled with fluid is related to the height h of the fluid above the hole. The relationship is given by:

$$v = \sqrt{2gh}$$

where g is the acceleration due to gravity and is value is $9.8\ m/s^2$. Assuming the fluid is draining from the tank, and the height of the fluid $h(t)$ decreases over time, the rate of change of h with respect to time t can be modeled by the first-order differential equation:

$$\frac{dh(t)}{dt} = -\frac{a}{A}\sqrt{2gh}(t)$$

where A is the cross-sectional area of the tank and its value is $10\ m^2$, and a is the area of the hole and its value is $1\ m^2$. Given an initial height of the fluid $h(0)$, find the height $h(t)$ of the fluid as a function of time t.

[16 marks]

(b) As fluid exiting through the hole, with initial fluid height $h(0) = 20m$, determine the fluid height after $60\ s$?

[4 marks]

Total 20 Marks

Part 2:

(a) A capacitor in an RC circuit is discharging through a resistor. The voltage across the capacitor, $v(t)$ which is a function of time t, can be modeled by the following differential equation:

$$\frac{dv(t)}{dt} = -\frac{1}{RC}v(t)$$

where R is the resistance in Ω, C is the capacitance in F, and $v(t)$ is the voltage across the capacitor in V.

Assume $R = 10\ \Omega$, $C = 1\ F$, and the initial voltage across the capacitor is $v(0)\ V$. Solve the differential equation to find the voltage $v(t)$ as a function of time t.

[16 Marks]

(b) Assume the initial voltage across the capacitor is $v(0) = 20\ V$. Calculate the voltage $v(t)$ after $5\ s$.

[4 Marks]

Total 20 Marks

PLEASE TURN THE PAGE…

Q4: Laplace Transform-Mechanical

Solve one of the two parts below:

Part 1:

(a) The equation of motion for the object is given by Stokes' Law, which relates the drag force experienced by the object to its velocity. The ordinary differential equation (ODE) for the velocity $v(t)$ of the object as a function of time t can be described by:

$$v'(t) = g - \beta v(t)$$

where:

- g is the acceleration due to gravity (assumed to be $9.8\, m/s^2$ for this question).
- β is a constant that represents the viscous damping coefficient per unit mass (assumed to be $0.1\, s^{-1}$ for this question).

Given that the object is released from rest at $t = 0$ ($v(0) = 0\, m/s$),use the method of Laplace transforms to solve for the velocity $v(t)$ as a function of time t.

[16 marks]

(b) Estimate the velocity $v(t)$ after $10\, s$.

[4 marks]
Total 20 Marks

Part 2:

(a) An RC (resistor-capacitor) circuit can be described by the following first-order differential equation when a constant voltage source is applied:

$$V_R{}'(t) + \frac{1}{RC} V_R(t) = \frac{V_s}{RC}$$

where:

- $V_R(t)$ is the voltage across the resistor at time t.
- R is the resistance in Ω.
- C is the capacitance in F.
- V_s is the constant supply voltage.

Given the following initial condition: $V_R(0) = 0$ when $t = 0\, s$ (i.e., the capacitor is uncharged at the beginning), and a supply voltage $V_s = 12\, V$, resistor $R = 2\, \Omega$, and capacitor $C = 1\, F$.
Use the method of Laplace transform to derive an expression for $V_R(t)$.

[16 marks]

(b) Estimate the voltage across the resistor, $V_R(t)$ after $10\, s$. **[4 marks]**

Total 20 Marks

PLEASE TURN THE PAGE…

Q5: Fourier transform

A square wave signal is a fundamental type of signal in electronics, especially in digital electronics and pulse modulation techniques. The signal can be defined as:

$$g(t) = \quad 3; \qquad for \ -5 \le t \le 0$$

$$g(t) = -3; \qquad for \quad 0 \le t \le 5$$

$$g(t) = \quad 0; \qquad for \qquad |t| > 5$$

(a) Sketch the signal waveform from the equations and comment on the result.

[6 Marks]

(b) The Fourier Transform decomposes the square wave into its constituent frequencies. In signal processing, understanding the frequency components of a signal like a square wave helps in noise reduction, signal reconstruction, and in the development of efficient transmission and encoding schemes. Calculate the Fourier transform $G(\omega)$ of the signal waveform and comment on the result.

[14 Marks]

Total 20 Marks

PLEASE TURN THE PAGE…

Q6: Matrices

<u>Solve one of the two parts below:</u>

Part 1:

In an electrical system, consider a network of resistors and inductors arranged in a certain configuration. The dynamic behaviour of the system can be modelled using the state-space representation, where the state vector x represents electrical quantities like current and voltage across specific components.

The state-space representation of the system is given by:

$$\dot{x} = Ax$$

Where

$$x = \begin{Bmatrix} I_1 \\ V_1 \end{Bmatrix}$$

And the system matrix A is given as

$$A = \begin{pmatrix} -2 & 5 \\ 3 & -4 \end{pmatrix}$$

Here, I_1 represents the current through a particular component (like an inductor), and V_1 represents the voltage across another component (like a resistor).

a) Calculate the eigenvalues of the matrix A. These eigenvalues represent key characteristics of the system's dynamic response.

[08 marks]

b) Determine the eigenvectors corresponding to these eigenvalues. These eigenvectors represent the modes of the system's behaviour.

[12 marks]

Total 20 marks

Question 6 (part 2) continues over the page…

Question 6 continued…

Part 2:

Consider a two-dimensional deformation scenario where the strain ($\boldsymbol{m\varepsilon}$) tensor (matrix) at a specific point in a metal beam is given by:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 2 & -0.5 \\ -0.5 & 3 \end{pmatrix}$$

This tensor represents the strain in the beam under certain loading conditions, measured in units of deformation per unit length.

The eigenvalues of the stress tensor (matrix) give the principal stresses and the eigenvectors give their principal directions.

(a) Calculate the eigenvalues of the strain tensor. These eigenvalues represent the principal strains of the beam at the given point.

[8 marks]

(b) Determine the eigenvectors corresponding to these eigenvalues. These eigenvectors indicate the directions of the principal strains.

[12 marks]

Total 20 marks

PLEASE TURN THE PAGE…

Q7: Simpson's rule

Solve one of the two parts below:

Part 1:

During a test session on a new model of an electric train, the acceleration (a) in meters per second squared (m/s^2) is recorded at different time intervals to understand the train's performance during a full-speed run. The table below presents the recorded acceleration (a) of the train over a one-minute interval, sampled every five seconds. To calculate the velocity of the train, it is assumed that the train starts from rest (initial velocity is $0\ m/s$), and the velocity at each time interval can be estimated by the product of acceleration and time $(a * t)$. Use these values to estimate the train's final velocity.

Time - t (s)	0	5	10	15	20	25	30	35	40	45	50	55	60
Acceleration - a (m/s^2)	0	2.2	3.5	4.0	4.5	4.2	3.8	3.0	2.5	2.0	1.5	1.0	0.5

(a) Plot the graph of acceleration (a) versus time (t) using the data given in the table and label the graph clearly.

[6 marks]

(b) Using Simpson's rule, estimate the train's final velocity at the end of the $1\ min$ interval. Note that the velocity at any time can be approximated as the integral of acceleration with respect to time.

[14 marks]

Total 20 marks

Question 7 (part 2) continues over the page…

PLEASE TURN THE PAGE…

Question 7 continued…

Part 2:

In a renewable energy study, the output power of a solar panel is analysed throughout a sunny day. The power output (P) in watts (W) is recorded at specific time intervals over ten hours to evaluate the energy production capacity. The table below presents the power output of the solar panel at each hour from sunrise to sunset. The goal is to estimate the total energy produced (in Wh) by the solar panel over this period.

Time-t (h)	0	1	2	3	4	5	6	7	8	9	10
Power-$P(W)$	0	20	55	105	180	260	320	350	355	300	190

(a) Construct a graph of the power output (P) versus the time (t) using the data given in the table and ensure the graph is clearly labelled.

[6 marks]

(b) Utilise Simpson's rule to estimate the total energy produced by the solar panel throughout the day. Remember that the energy produced can be approximated as the integral of power with respect to time.

[14 marks]

Total 20 marks

PLEASE TURN THE PAGE…

Q8) Partial derivative and double integrals

(a) An engineer is analysing the thermal performance of a resistor. The temperature T $(°_C)$ on the surface of the resistor depends on time $t(s)$ and the position x (m) along the length of the resistor. The temperature distribution is given by the function:

$$T(x,t) = e^{-t} \cdot \sin(\pi x)$$

Determine the rate of change of temperature with respect to time and position by calculating the expression E defined as:

$$E = \frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial x \partial t}$$

Evaluate E at the point where $x = 0.5\ m$ and $t = 2\ s$

[10 marks]

(b) An engineer is designing a planar coil for an inductor. The magnetic field B generated by the coil at different points in the xy-plane is given by the function $B(x,y) = xy^2$ in T. The coil extends over a rectangular area defined by the corners $(1,1)$, $(1,4)$, $(4,1)$, and $(4,4)$. Calculate the total magnetic flux, $\emptyset$ through this area using a double integral.

Evaluate the double integral:

$$\emptyset = \int_{x=1}^{x=4} \int_{y=1}^{y=4} xy^2 dydx$$

Express your answer in Wb (the SI unit of magnetic flux).

[10 marks]

Total 20 marks

END OF QUESTIONS

FORMULA SHEETS FOLLOW OVER THE PAGE….

School of Engineering
BEng (Hons) Mechanical Engineering
BEng (Hons) Electrical and Electronics Engineering
Semester 1 Examination 2023/2024
Engineering Modelling and Analysis
Module No. AME5014

Formula sheet

Partial Fractions

$$\frac{F(x)}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\frac{F(x)}{(x+a)(x+b)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2}$$

$$\frac{F(x)}{(x^2+a)} = \frac{Ax+B}{(x^2+a)}$$

Small Changes

z = f(u, v, w)

$$\delta z \simeq \frac{\partial z}{\partial u}.\delta u + \frac{\partial z}{\partial v}.\delta v + \frac{\partial z}{\partial w}.\delta w$$

Total Differential

z = f(u, v, w)

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

Rate of Change

z = f(u, v, w)

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}.\frac{du}{dt} + \frac{\partial z}{\partial v}.\frac{dv}{dt} + \frac{\partial z}{\partial w}.\frac{dw}{dt}$$

Eigenvalues

$$|A - \lambda \mathrm{I}| = 0$$

PLEASE TURN THE PAGE…

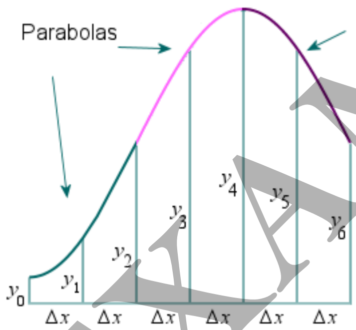
Eigenvectors

$$(A-\lambda_r \mathrm{I})x_r = 0$$

Integration

$$\int u.\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}dx$$

Simpson's rule
To calculate the area under the curve which is the integral of the function **Simpson's Rule** is used as shown in the figure below:



The area into *n* equal segments of width *Δx*. Note that in Simpson's Rule, *n* must be EVEN. The approximate area is given by the following rule:

$$Area = \int_a^b f(x)dx = \frac{\Delta x}{3}(y_0 + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots) + y_n)$$

Where $\Delta x = \frac{b-a}{n}$

Differential equation

Homogeneous form:

$$a\ddot{y} + b\dot{y} + cy = 0$$

Characteristic equation:

$$a\lambda^2 + b\lambda + c = 0$$

Quadratic solutions :

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i. If $b^2 - 4ac > 0$, λ_1 and λ_2 are distinct real numbers then the general solution of the differential equation is:

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

A and B are constants.

ii. If $b^2 - 4ac = 0$, $\lambda_1 = \lambda_2 = \lambda$ then the general solution of the differential equation is:

$$y(t) = e^{\lambda t}(A + Bx)$$

A and B are constants.

iii. If $b^2 - 4ac < 0$, λ_1 and λ_2 are complex numbers then the general solution of the differential equation is:

$$y(t) = e^{\alpha t}[Acos(\beta t) + Bsin(\beta t)]$$

$$\alpha = \frac{-b}{2a} \quad and \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

A and B are constants.

Inverse of 2x2 matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of A can be found using the formula:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>modelling growth and decay of engineering problem</u>

$C(t) = C_0 e^{kt}$

$k > 0$ gives exponential growth

$k < 0$ gives exponential decay

<u>First order system</u>

$$y(t) = \ k(1 - e^{-\frac{t}{\tau}})$$

Transfer function: $\frac{k}{\tau s+1}$

Derivatives table:

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, any constant	0
x	1
x^2	$2x$
x^3	$3x^2$
x^n, any constant n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\operatorname{cosec} x = \frac{1}{\sin x}$	$-\operatorname{cosec} x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$

PLEASE TURN THE PAGE…

Integral table:

$f(x)$	$\int f(x)\,\mathrm{d}x$
k, any constant	$kx + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$\mathrm{e}^x + c$
e^{kx}	$\frac{1}{k}\mathrm{e}^{kx} + c$
$\cos x$	$\sin x + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sin x$	$-\cos x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\tan x$	$\ln(\sec x) + c$
$\sec x$	$\ln(\sec x + \tan x) + c$
$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x) +$
$\cot x$	$\ln(\sin x) + c$
$\cosh x$	$\sinh x + c$
$\sinh x$	$\cosh x + c$
$\tanh x$	$\ln\cosh x + c$
$\coth x$	$\ln\sinh x + c$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$

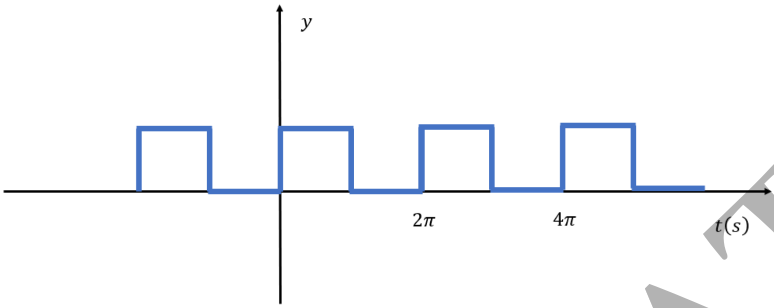
PLEASE TURN THE PAGE…

Laplace table:

$f(t)$	$F(s)$		$f(t)$	$F(s)$
1	$\frac{1}{s}$		$u_c(t)$	$\frac{e^{-cs}}{s}$
t	$\frac{1}{s^2}$		$\delta(t)$	1
t^n	$\frac{n!}{s^{n+1}}$		$\delta(t-c)$	e^{-cs}
e^{at}	$\frac{1}{s-a}$		$f'(t)$	$sF(s)-f(0)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$		$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$\cos bt$	$\frac{s}{s^2+b^2}$		$(-t)^n f(t)$	$F^{(n)}(s)$
$\sin bt$	$\frac{b}{s^2+b^2}$		$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$		$e^{ct}f(t)$	$F(s-c)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$		$\delta(t-c)f(t)$	$e^{-cs}f(c)$

Fourier Series

The periodic square wave with Fourier Series and the coefficients of the Fourier Series



The function which represent the periodic square wave can be represented by

$$y = f(t)$$

Period of the function:

$$T = 2\pi \frac{sec}{cycle}$$

Fourier series of the function:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \cdots + a_n \cos(nt)$$
$$+ b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \cdots b_n \sin(nt)$$

Where, $n = 1,2,3,4,5, \dots \dots$

Alternatively,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

Fourier Coefficients:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(nt)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(nt)\, dt$$

PLEASE TURN THE PAGE…

Useful Equations for Fourier transform

Fourier transform equation

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

Inverse Fourier transform equation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

Euler's formula for trigonometric identities

$$e^{j\theta} = \sin\theta + j\cos\theta$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

Where, $j = \sqrt{-1}$

For any arbitrary function

$$\int_{a}^{b} f(t)\delta(t - t_0)\, dt = f(t_0)$$

<u>END OF PAPER</u>