[ENG07]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING SCIENCES

BEng (HONS) MECHANICAL, ELECTRICAL & ELECTRONIC ENGINEERING

SEMESTER ONE EXAMINATION 2021/22

ENGINEERING MODELLING AND ANALYSIS

MODULE NO: AME5014

Date: Wednesday 12th January 2022 Time: 14:00 – 16:00

INSTRUCTIONS TO CANDIDATES:

There are EIGHT questions.

Answer ANY FIVE questions.

All questions carry equal marks.

Marks for parts of questions are shown in brackets.

Electronic calculators may be used if data and program storage memory is cleared prior to the examination.

CANDIDATES REQUIRE:

Formula Sheets (attached following questions).

Q1: Differentiation-Integration

(a) Faraday's law states that the electromotive force, E in volt (V), induced by N turns of a coil which a flux, Φ, passing through it, is given by

$$E = -N \frac{d\,\Phi}{dt}$$

If $N = 100$ and $\Phi = 10sin(120\pi t)$, determine E.

(10 Marks)

(b) The displacement, s in m of an object is given by

$$s = \int_0^3 (t^4 + t)dt$$

Where t is time in sec. Evaluate s in m.

(10 Marks)

Total 20 Marks

Q2: Second Order Differential Equation

Solve ONE of the TWO parts below:

Part 1:

(a) A mass-spring-damper system can be modelled by the following equation:

$$y(t)'' - 5y(t)' + 6y(t) = 0$$

For which y is the displacement in function of the time (t). The initial conditions are $y(0) = 0m$ and $y'(0) = 2\ m/s$.

Solve analytically the equation above to find the displacement, $y(t)$, and state the nature of displacement response.

(14 Marks)

(b) Create a table listing displacement, $y(t)$, found in (a) for $t = 1,2,3\ (sec)$.

(6 Marks)

Total 20 Marks

Part 2:

(a) An RCL circuit in series, driven by a constant emf, can be modelled by the following equation:

$$i(t)'' + 4i(t)' + 4i(t) = 0$$

For which the natural response of the circuit is i, the current in function of the time (t). The initial conditions are $i(0) = 2\ A$ and $i'(0) = 4A/s$.

Solve analytically the equation above to find the current, $i(t)$, and state the nature of response of the current.

(14 Marks)

(b) Create a table listing the current, $i(t)$ found in (a) for $t = 1,2,3\ (sec)$.

(6 Marks)

Total 20 Marks

PLEASE TURN THE PAGE…

Q3: First Order Differential Equation

Solve ONE of the TWO parts below:

Part 1:

(a) According to the **drag equation,** the velocity (v) of an object moving through a fluid can be modelled by the following equation:

$$\frac{dv}{dt} = -kv^2$$

Where k is a constant and v is the velocity of the object in function of time (t).

Find the general solution of the above equation giving the velocity v in function of time (t).

(12 Marks)

(b) An object moving through the water has an initial velocity of $40\ m/s$. Two seconds later, the velocity has decreased to $30\ m/s$. What will be the fully defined equation of its velocity in function of time (t).

(8 Marks)

Total 20 Marks

Part 2:

(a) A mobile phone battery has a voltage, v. Its recharging is modelled by the following equation.

$$\frac{dv}{dt} = 2(20 - v)$$

Find the expression for v in terms of t.

(12 Marks)

(b) In the beginning of the time ($t = 0\ sec$), the voltage, v, was $0V$. Estimate the time taken for the voltage, v to reach $10V$ of charge.

(8 Marks)

Total 20 Marks

PLEASE TURN THE PAGE…

Q4: Laplace Transforms

Solve <u>ONE</u> of the <u>TWO</u> parts below:

Part 1:

(a) The ordinary differential equation (ODE) describing the temperature $T(t)$ of an object placed in an environment at $2\,^{\circ}C$ can be modelled approximately by the Newton cooling equation below with $T(t)$ representing temperature at time t in seconds.

$$\frac{dT(t)}{dt} = -4 \times 10^{-4}(T(t) - 2)$$

Given the object is initially ($t = 0\ s$) placed into a refrigerator at $20\,^{\circ}C$ till equilibrium.

Use the method of **Laplace transforms** to derive an expression for $T(t)$.

(12 Marks)

(b) Estimate the time t in $minutes$ taken for the object to cool down to $7\,^{\circ}C$ in the refrigerator with an environment at $2\,^{\circ}C$.

(8 Marks)

Total 20 Marks

Part 2:

(a) An electronics unit can be modelled using the following equation:

$$v_c' + 2v_c = 60$$

Where v_C is the voltage of the capacitor, which changes with time t in seconds.

Given: initial condition, $v_c(0) = 0$ when $t = 0\ s$.

Use the method of **Laplace transforms** to derive an expression for $v_C(t)$.

(12 Marks)

(b) Estimate the time t (in sec) taken for the voltage of the capacitor, $v_C(t)$ to reach $8\,V$.

(8 Marks)

Total 20 Marks

PLEASE TURN THE PAGE…

Q5: Fourier transform

An electronic/mechanical signal can be modelled by the following equations in the time domain t.

$$
\begin{aligned}
f(t) &= \;\;5; & for\ -3 \le t \le 0 \\
f(t) &= -5; & for\quad 0 \le t \le 3 \\
f(t) &= \;\;0; & for\qquad |t| > 3
\end{aligned}
$$

(a) Sketch the signal waveform from the equations and comment on the result.

(6 Marks)

(b) Calculate the Fourier transform $F(\omega)$ of the signal waveform and comment on the result.

(14 Marks)

Total 20 Marks

Q6: Matrices

Solve ONE of the TWO parts below:

Part 1:

For the electrical circuit shown in Figure 6 below.

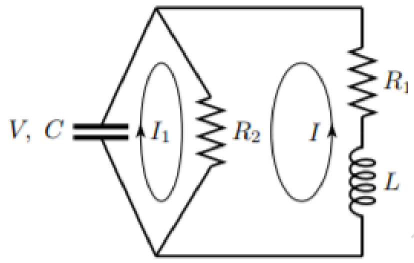


Figure 6: Electrical circuit.

The motions equations are as follows:

$$\dot{x} = Ax$$

Where

$$x = \begin{Bmatrix} I \\ V \end{Bmatrix}$$

If the matrix is given as

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

a) Find the eigenvalues of matrix A.

(8 Marks)

b) Find the eigenvectors of matrix A.

(12 Marks)

Total 20 Marks

Question 6 continues over the page….

PLEASE TURN THE PAGE…

Question 6 continued….

Part 2:

The stress tensor (matrix) below can define a two-dimensional stress analysis in MPa of a supporting plate at a point as follows:

$$\boldsymbol{\sigma} = \begin{pmatrix} 50 & -20 \\ -20 & 80 \end{pmatrix}$$

The eigenvalues of the stress tensor (matrix) give the principal stresses and the eigenvectors give their principal directions.

(a) Find the principal stresses.

(8 Marks)

(b) Find the principal directions.

(12 Marks)

Total 20 marks

PLEASE TURN THE PAGE….

Q7: Simpson's rule

Solve <u>ONE</u> of the <u>TWO</u> parts below:

Part 1:

In a car race, the velocity (v) of a car in meters per second (m/s) and the corresponded time instants (t) in second (s) are measured thirteen times to get the distance travelled $(v * t)$. The table below gives the recorded velocity (v) of the car and the corresponding time instants (t).

Time - t (s)	0	5	10	15	20	25	30	35	40	45	50	55	60
Velocity - v (m/s)	32	38	33	40	44	28	35	29	41	42	38	40	36

(a) Sketch the graph of velocity (v) versus time (t) from the data given in the table and annotate the graph appropriately.

(6 Marks)

(b) Find the total distance travelled by the car $(v * t)$ using Simpson's rule.

(14 Marks)

Total 20 Marks

Part 2:

In an electrical Lab, an experiment is conducted and the voltage (V) and the current (I) are measured eleven times to get the electrical power$(V * I)$. The table below gives the recorded values of the voltage (V) and the current (I).

Voltage (V) (V)	0	2	4	6	8	10	12	14	16	18	20
Current (I) (A)	0	6.5	14.9	22.3	29.8	38.3	47.1	54.7	61.2	68.4	75.9

(a) Sketch the graph of the voltage (V) versus the current (I) from the data given in the table and annotate the graph appropriately.

(6 Marks)

(b) Find the total power in W used in the experiment using Simpson's rule.

(**14 marks**)

Total 20 Marks

PLEASE TURN THE PAGE…

Q8: Partial derivative and double integrals

(a) If f is given as

$$f = e^{y} * \ln(x)$$

Evaluate Z, so that:

$$Z = \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x \partial y}$$

If $x = 3$ and $y = -2$.

(**10 Marks**)

(b) Evaluate the following double integrals

$$\int_{x=1}^{x=4} \int_{y=2}^{y=5} (2\,x^3 - 3\,y^4)\,dy\,dx$$

(**10 Marks**)

Total 20 Marks

<u>**END OF QUESTIONS**</u>

FORMULA SHEET FOLLOWS ON NEXT PAGES

PLEASE TURN THE PAGE…

Formula sheet

Partial Fractions

$$\frac{F(x)}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\frac{F(x)}{(x+a)(x+b)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2}$$

$$\frac{F(x)}{(x^2+a)} = \frac{Ax+B}{(x^2+a)}$$

Small Changes

z = f(u, v, w)

$$\delta z \simeq \frac{\partial z}{\partial u}.\delta u + \frac{\partial z}{\partial v}.\delta v + \frac{\partial z}{\partial w}.\delta w$$

Total Differential

z = f(u, v, w)

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw$$

Rate of Change

z = f(u, v, w)

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}.\frac{du}{dt} + \frac{\partial z}{\partial v}.\frac{dv}{dt} + \frac{\partial z}{\partial w}.\frac{dw}{dt}$$

Eigenvalues

$$|A - \lambda \mathrm{I}| = 0$$

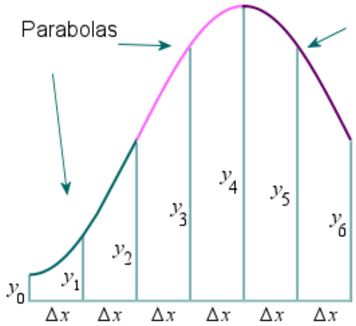
Eigenvectors

$$(A - \lambda_r \mathrm{I})x_r = 0$$

Integration

$$\int u.\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}dx$$

Simpson's rule
To calculate the area under the curve which is the integral of the function **Simpson's Rule** is used as shown in the figure below:



The area into *n* equal segments of width *Δx*. Note that in Simpson's Rule, *n* must be EVEN. The approximate area is given by the following rule:

$$Area = \int_a^b f(x)dx = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 \dots + 4y_{n-1} + y_n)$$

Where $\Delta x = \frac{b-a}{n}$

<u>Differential equation</u>

Homogeneous form:

$$a\ddot{y} + b\dot{y} + cy = 0$$

Characteristic equation:

$$a\lambda^2 + b\lambda + c = 0$$

Quadratic solutions :

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i. If $b^2 - 4ac > 0$, λ_1 and λ_2 are distinct real numbers then the general solution of the differential equation is:

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

A and B are constants.

ii. If $b^2 - 4ac = 0$, $\lambda_1 = \lambda_2 = \lambda$ then the general solution of the differential equation is:

$$y(t) = e^{\lambda t}(A + Bx)$$

A and B are constants.

iii. If $b^2 - 4ac < 0$, λ_1 and λ_2 are complex numbers then the general solution of the differential equation is:

$$y(t) = e^{\alpha t}[Acos(\beta t) + Bsin(\beta t)]$$

$$\alpha = \frac{-b}{2a} \quad and \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

A and B are constants.

<u>Inverse of 2x2 matrices:</u>

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of A can be found using the formula:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

PLEASE TURN THE PAGE…

Modelling growth and decay of engineering problem

$C(t) = C_0 e^{kt}$

$k > 0$ gives exponential growth

$k < 0$ gives exponential decay

First order system

$$y(t) = k(1 - e^{-\frac{t}{\tau}})$$

Transfer function: $$\frac{k}{\tau s+1}$$

Derivatives table:

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, any constant	0
x	1
x^2	$2x$
x^3	$3x^2$
x^n, any constant n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\operatorname{cosec} x = \frac{1}{\sin x}$	$-\operatorname{cosec} x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$

Integral table:

$f(x)$	$\int f(x)\,\mathrm{d}x$
k, any constant	$kx + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$\mathrm{e}^x + c$
e^{kx}	$\frac{1}{k}\mathrm{e}^{kx} + c$
$\cos x$	$\sin x + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sin x$	$-\cos x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\tan x$	$\ln(\sec x) + c$
$\sec x$	$\ln(\sec x + \tan x) + c$
$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x) +$
$\cot x$	$\ln(\sin x) + c$
$\cosh x$	$\sinh x + c$
$\sinh x$	$\cosh x + c$
$\tanh x$	$\ln\cosh x + c$
$\coth x$	$\ln\sinh x + c$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$

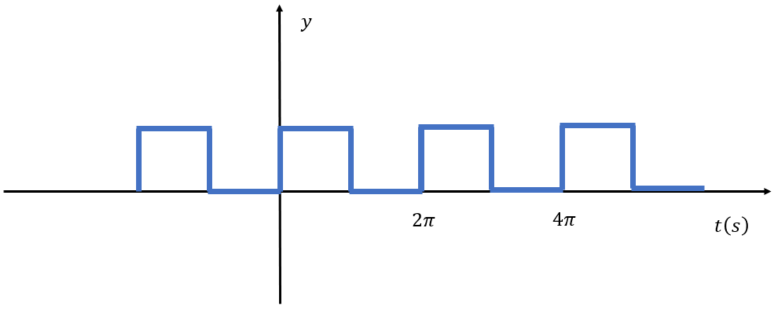
PLEASE TURN THE PAGE…

Laplace table:

$f(t)$	$F(s)$		$f(t)$	$F(s)$
1	$\frac{1}{s}$		$u_c(t)$	$\frac{e^{-cs}}{s}$
t	$\frac{1}{s^2}$		$\delta(t)$	1
t^n	$\frac{n!}{s^{n+1}}$		$\delta(t-c)$	e^{-cs}
e^{at}	$\frac{1}{s-a}$		$f'(t)$	$sF(s)-f(0)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$		$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$\cos bt$	$\frac{s}{s^2+b^2}$		$(-t)^n f(t)$	$F^{(n)}(s)$
$\sin bt$	$\frac{b}{s^2+b^2}$		$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$		$e^{ct}f(t)$	$F(s-c)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$		$\delta(t-c)f(t)$	$e^{-cs}f(c)$

Fourier Series

The periodic square wave with Fourier Series and the coefficients of the Fourier Series



The function which represent the periodic square wave can be represented by

$$y = f(t)$$

Period of the function:

$$T = 2\pi \frac{sec}{cycle}$$

Fourier series of the function:

$$f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + \cdots + a_n\cos(nt)$$
$$+b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + \cdots b_n\sin(nt)$$

Where, $n = 1,2,3,4,5, \ldots \ldots ..$

Alternatively,

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt))$$

Fourier Coefficients:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(nt)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(nt)\,dt$$

PLEASE TURN THE PAGE…

Useful Equations for Fourier transform

Fourier transform equation

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

Inverse Fourier transform equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

Euler's formula for trigonometric identities

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

Where, $j = \sqrt{-1}$

For any arbitrary function

$$\int_a^b f(t)\delta(t - t_0)\, dt = f(t_0)$$

End of the Formula Sheet

<u>END OF PAPER</u>