[ENG-17]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

B.Sc. (Hons) MATHEMATICS

SEMESTER 2 EXAMINATIONS 2021/22

FLUID DYNAMICS

MODULE NO: MMA6003

Date: Tuesday 17th May 2022 **Time: 14.00 – 16.15**

INSTRUCTIONS TO CANDIDATES:

1. **Answer ALL FOUR questions.**
2. **All questions carry EQUAL marks.**
3. **Maximum marks for each part/question are shown in brackets.**

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1. (a) A liquid of constant density ρ is in steady, irrotational motion and is subject to a conservative body force $\boldsymbol{F}$ with potential Ω. From the equation of motion,

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}.\nabla)\boldsymbol{q} = -\frac{1}{\rho}\nabla p + \boldsymbol{F},$$

establish Bernoulli's equation:

$$\frac{1}{2}q^2 + \frac{p}{\rho} - \Omega = \text{constant}$$

where p is the pressure and $\boldsymbol{q}$ is the velocity at any point in the liquid.

(10 marks)

(b) A cylindrical reservoir of liquid has radius A and contains liquid to a depth h_0. Liquid issues from the reservoir through a circular hole in the bed of radius a. The flow is steady and irrotational.

(i) Use Bernoulli's equation to show that

$$q_s^2 = q_h^2 - 2gh$$

where q_s = the speed of the liquid at the surface;
q_h = the speed of the liquid issuing from the hole;
$h \equiv h(t)$ =the height of the surface at time t.

(5 marks)

(ii) Use mass-conservation to show that the equation in part (i) above can be written as

$$q_h = A^2\left(\frac{2gh}{A^4 - a^4}\right)^{1/2}.$$

(4 marks)

(iii) Show that the time taken for the reservoir to drain is

$$\frac{1}{a^2}\left(\frac{2h_0}{g}(A^4 - a^4)\right)^{\frac{1}{2}}.$$

(6 marks)

2. A steady, irrotational flow of inviscid liquid has velocity potential

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta,$$

where U is a constant, in the usual notation of cylindrical polar coordinates.

(i) Show that the potential function satisfies Laplace's equation and find the velocity field.

(6 marks)

(ii) Show that the boundary condition on an infinite circular cylinder of radius a immersed in the fluid with its axis along the z-axis is satisfied by this velocity field.

(2 marks)

(iii) Find the velocity of the liquid as a vector in Cartesian coordinates at large distances from the cylinder.

(5 marks)

(iv) There are no external body forces acting on the liquid and the pressure is p_∞ as $r \to \infty$. Show that the pressure on the cylinder is

$$p_\infty + \tfrac{1}{2}\rho U^2(1 - 4\sin^2\theta)$$

where ρ is the uniform density of the liquid. Bernoulli's equation may be used.

(8 marks)

(v) Show that p_∞ must not be less than $\dfrac{3\rho U^2}{2}$ to avoid cavitation.

(4 marks)

3. Viscous fluid flows steadily along a long straight pipe with vertical generators lying parallel to the z-axis.

(a) Show that a velocity field of the form

$$\boldsymbol{u} = w(x, y)\boldsymbol{k}$$

satisfies:

(i) mass-conservation; (3 marks)

(ii) the Navier-Stokes equations provided that

$$\nabla^2 w = A$$

and write down the value of A explaining carefully why it is a constant. You may use the relation $\nu = \mu/\rho$.

(10 marks)

(b) Find $w(x, y)$ if the pipe has elliptical section

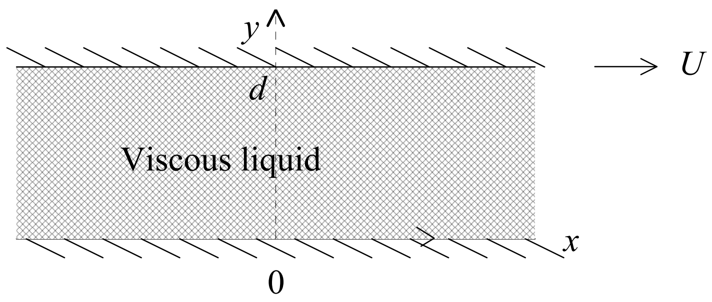
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(5 marks)

(c) Find the mass-flux per unit time through the pipe. You may find the change of variables $x = ar\cos\theta, \quad y = br\sin\theta$ helpful in the double integral used to evaluate the mass-flux.

(7 marks)

4. Liquid of kinematic viscosity ν is bounded by two infinite parallel plates at $y = 0$ and $y = d$. Initially, the fluid is at rest and there is no applied pressure gradient. The upper plate at $y = d$ is suddenly jerked into motion with constant velocity $U\boldsymbol{i}$. The diagram below illustrates the flow boundaries.



(a) Show that a velocity field of the form

$$\boldsymbol{u} = u(y, t)\boldsymbol{i}$$

Satisfies:

(i) the mass-conservation equation; (2 marks)

(ii) the Navier-Stokes equations provided that $u(y, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

and write down the associated initial and boundary conditions. (5 marks)

(b) Obtain the steady-state solution. (5 marks)

(c) Use the method of separation of variables to obtain the full unsteady solution. (13 marks)

END OF QUESTIONS