[ESS02]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

BSc (Hons) MATHEMATICS

SEMESTER 1 EXAMINATIONS 2019/20

FURTHER LINEAR ALGEBRA

MODULE NO: MMA6002

Date: Tuesday 14th January 2020 **Time:10.00-12.15**

INSTRUCTIONS TO CANDIDATES:

1. **Answer all FOUR questions.**
2. **All questions carry equal marks.**
3. **Maximum marks for each part/question are shown in brackets.**

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1. (a) In the general linear Lie algebra $gl(n,\boldsymbol{R})$ the product is given by

$$[A,B] = AB - BA$$

Show that this product is anti-commutative.

(2 marks)

Show that this product satisfies the Jacobi identity.

(10 marks)

(b) A real matrix is *skew-symmetric* if $A^T = -A$.

Let $o(n)$ be the set of $n \times n$ real skew-symmetric matrices.

Show that $o(n)$ is a subspace of $gl(n,\boldsymbol{R})$.

(5 marks)

Show that $o(n)$ is a Lie subalgebra of $gl(n,\boldsymbol{R})$.

(3 marks)

(c) Explain carefully why the dimension of $o(n)$ as a real vector space is given by

$$\dim o(n) = \frac{1}{2}n(n-1).$$

(5 marks)

Please turn the page

2. (a) Let $\phi : G \to H$ be a smooth homomorphism of matrix groups, and let TG and TH be the tangent spaces of G and H respectively.

We may define a mapping $T_\phi : TG \to TH$ by

$$T_\phi(\gamma'(0)) = (\phi \circ \gamma)'(0)$$

Show that T_ϕ is a linear transformation.

(12 marks)

(b) Let A be an $n \times n$ real matrix with trace 0.

Show that A belongs to the tangent space of the special linear group $SL(n, \boldsymbol{R})$.

You may use the fact that $\det \exp A = e^{\operatorname{tr} A}$ for any square matrix A.

(5 marks)

(c) Explain what is meant by the *centre* of a group G.

Show that the centre of G is a subgroup of G.

Find the centre of the special unitary group $SU(5)$.

You may assume that all matrices in the centre are scalar matrices.

(8 marks)

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3. (a) Let A be a normed real algebra.

Prove that $\langle zx, y\rangle = \langle x, \bar{z}y\rangle$ for all $x, y, z \in A$.

You may use the fact that $\langle zx, zy\rangle = \langle z, z\rangle\langle x, y\rangle$ for all $x, y, z \in A$.

(8 marks)

(b) For the quaternions $q_1 = 3 + i - 2j + k$ and $q_2 = 1 - 3i + 2j + k$ calculate

(i) $q_1 + q_2$ (ii) $q_1 q_2$

(iii) $q_2 q_1$ (iv) q_1^{-1}

(9 marks)

(c) By identifying the quaternions $\boldsymbol{H}$ with $\boldsymbol{R}^4$ we may define a linear transformation $\boldsymbol{R}^4 \to \boldsymbol{R}^4$ by multiplication by a quaternion.

Find the matrix that represents the linear transformation $f : \boldsymbol{R}^4 \to \boldsymbol{R}^4$ given by $f(v) = vq$, where $q = 1 + 3i + 5j + 7k$.

(8 marks)

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4. (a) Define a mapping $\phi : \boldsymbol{H} \otimes Mat(2, \boldsymbol{R}) \to Mat(2, \boldsymbol{H})$ by

$$\phi(q \otimes A) = qA$$

Show that ϕ is an algebra map.

By considering $\ker \phi$ show that ϕ is injective.

State the dimension of each of $\boldsymbol{H}$, $Mat(2, \boldsymbol{R})$ and $Mat(2, \boldsymbol{H})$.

Explain why it follows that ϕ is surjective, and hence an isomorphism.

(8 marks)

(b) The algebra $Cl'(2)$ has basis $\{1, e_1, e_2, e_1e_2\}$, where $e_1^2 = e_2^2 = 1$ and $e_2e_1 = -e_1e_2$.

Show that $(e_1e_2)^2 = -1$.

By choosing a suitable basis for $Mat(2, \boldsymbol{R})$ show that this algebra is isomorphic to $Cl'(2)$.

Hence determine the structure of $Cl(4)$.

You may use the following:

$$Cl(n+2) \cong Cl'(n) \otimes \boldsymbol{H}$$

(11 marks)

(c) Determine the structures of the Clifford algebras $Cl(12)$ and $Cl(20)$.

(6 marks)

END OF QUESTIONS