[ESS01]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

BSc (Hons) MATHEMATICS

SEMESTER 1 EXAMINATIONS 2019/20

ABSTRACT ALGEBRA

MODULE NO: MMA4001

Date: Tuesday 14th January 2020 **Time: 10.00-12.15**

INSTRUCTIONS TO CANDIDATES:

1. **Answer all FOUR questions.**
2. **All questions carry equal marks.**
3. **Maximum marks for each part/question are shown in brackets.**

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1. (a) Using induction prove that

$$1^3 + 2^3 + 3^3 + \ \dots \ + n^3 = \frac{1}{4}n^2(n+1)^2$$

for each natural number n.

(8 marks)

(b) If $P(S)$ is the power set of S, show that for a pair of sets S_1 and S_2 we have

$$P(S_1 \cap S_2) = P(S_1) \cap P(S_2)$$

(8 marks)

(c) Let S_1, S_2 and S_3 be sets.

Suppose $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are mappings. Define the composite mapping $g \circ f$, and state its domain and co-domain.

Show that if f and g are injective then $g \circ f$ is also injective.

(9 marks)

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2. (a) Construct the Cayley table for each of the groups $(\boldsymbol{Z}_7^*, \cdot)$ and $(\boldsymbol{Z}_6, +)$.

Show that these groups are isomorphic.

(10 marks)

(b) Suppose that $(\boldsymbol{G}, *)$ is a group and $H \subseteq G$. State necessary and sufficient conditions for H to be a subgroup of G.

Prove that if H_1 and H_2 are subgroups of G then $H_1 \cap H_2$ is also a subgroup of G.

(8 marks)

(c) Consider the mapping $f : (\boldsymbol{Z}, +) \to (\boldsymbol{Z}, +)$ given by $f(n) = 5n$.

Show that f is a homomorphism of groups. State the image, $\operatorname{im} f$. State the kernel $\ker f$, and say what can be deduced from this.

State, with reasons, whether or not the mapping $g : (\boldsymbol{Z}, +)_, \to (\boldsymbol{Z}, +)$ given by $f(n) = n + 5$ is a homomorphism.

(7 marks)

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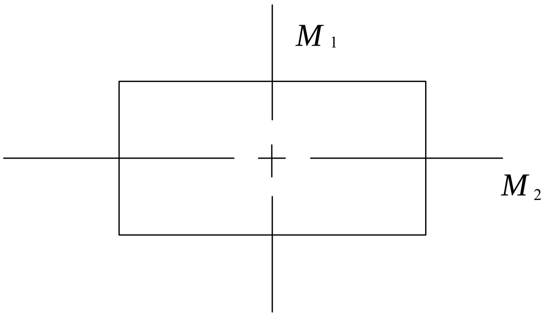
3. (a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 6 & 2 & 7 & 4 \end{pmatrix}$ and $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 7 & 5 & 2 & 1 & 4 \end{pmatrix}$ be permutations in S_7. Calculate the following permutations:

$$\sigma \circ \pi \qquad \pi \circ \sigma \qquad \sigma^3 \qquad \pi^{-1}$$

(8 marks)

(b) Consider a rectangle



Reflection in the line M_1 or in the line M_2 is a symmetry of the figure. Describe the other two symmetries of the figure.

Draw the Cayley table for the group of symmetries of the rectangle. State the inverse of each element.

(9 marks)

(c) Find the centraliser of the matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

in the general linear group $GL(2, \boldsymbol{Q})$.

(8 marks)

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4. (a) State what is meant by an *integral domain* and give *two* examples. Let D be an integral domain. Prove that for all $a, x, y \in D$ with $a \neq 0$

$$\text{if } a \cdot x = a \cdot y \text{ then } x = y.$$

(8 marks)

(b) Let $z_1 = 3 + 8i$ and $z_2 = 4 - 7i$ be complex numbers. Calculate $z_1 z_2$ and $\frac{z_1}{z_2}$. Express each of z_1 and z_2 in polar coordinates. Find z_1^4 in polar coordinates.

(9 marks)

(c) Use the formula $e^{i\theta} = \cos\theta + i\sin\theta$ to verify the following trigonometric identities:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

(5 marks)

(d) Solve the quadratic equation

$$x^2 - 6x + 34 = 0.$$

(3 marks)

END OF QUESTIONS