PAST EXAMINATION PAPER [ESS03]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

B. Sc. (Hons) MATHEMATICS

SEMESTER 2: EXAMINATION 2018/19

REAL ANALYSIS

MODULE NUMBER: MMA5006

Date: 24th May 2019 **Time:** 2.00pm – 4.15pm

INSTRUCTIONS TO CANDIDATES:

1. Answer all FOUR questions.
2. Each question is worth 25 marks. The maximum marks possible for each part is shown in brackets.
3. The examination is closed-book.
4. A formula sheet is provided.

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1. (a) Let f be the real function defined by:

$$f(x) = \frac{x}{x-1}.$$

(i) Show *from first principles* (i.e. using an ϵ, δ argument) that:

$$\lim_{x \to \infty} f(x) = 1$$ (5 marks)

(ii) Show that f is continuous on $(1, \infty)$. (5 marks)

(b) Use the Intermediate Value Theorem to show that the equation:

$$x^6 - 2x^3 - x^2 = 12$$

has a solution on the interval $[1, 2]$. (5 marks)

(c) Let f be the real function defined on $\mathbb{R}\backslash\{0\}$ by

$$f(x) = \frac{1}{x}$$

(i) Show that f is Lipschitz continuous on $[1, 2]$. (5 marks)

(ii) Let g be another real function which is Lipschitz continuous on $[1, 2]$ with Lipschitz constant L. Prove that the function $f + g$ is Lipschitz continuous on $[1, 2]$. (5 marks)

2. (a) Let f be the function defined by

$$\begin{aligned} f : [0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \sqrt{x}\,. \end{aligned}$$

(i) Show that f is differentiable on $(0,\infty)$ and find its derivative. (5 marks)

(ii) Show that f is continuous at $x = 0$ but *not* differentiable there. (5 marks)

(b) Let f be a real function with continuous derivative on $[a,b] \subset \mathbb{R}$. Use the Extreme Value Theorem and the Mean Value Theorem to prove that f is Lipschitz on $[a,b]$. (5 marks)

(c) Let f be the function defined by:

$$f(x) = \ln(1+x).$$

By applying Taylor's Theorem to f about $x = 0$, prove:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2).$$ (10 marks)

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3. (a) Let f be the real function defined by

$$f(x) \;=\; 2x+1.$$

Use a uniform partition to show that f is Riemann integrable on $[1,2]$ with

$$\int_{x=1}^{2} f(x)\,dx \;=\; 4.$$

(10 marks)

(b) Let f be continuous and Riemann integrable on $[a,b]$. Define $\alpha, \beta \in [a,b]$ to be the locations of the minimum and maximum of f respectively.

(i) Prove: $$f(\alpha)(b-a) \;\leq\; \int_{x=a}^{b} f(x)\,dx \;\leq\; f(\beta)(b-a)$$ (5 marks)

(ii) Use the previous result to show that there exists $c \in (a,b)$ such that

$$f(c) \;=\; \frac{1}{b-a}\int_{x=a}^{b} f(x)\,dx.$$

(4 marks)

(iii) Define the function F by:

$$F(x) \;=\; \int_{t=a}^{x} f(t)\,dt.$$

Use the previous result to show that F is differentiable on $[a,b]$ with $F'(x)=f(x)$. (6 marks)

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4. (a) In the following, you may assume:

$$\frac{d}{dx}\Big(\cos(x)\Big) = -\sin(x) \qquad \text{and} \qquad \frac{d}{dx}\Big(\sin(x)\Big) = \cos(x).$$

(i) For fixed y, define the function f by:

$$f(x) = \cos(x+y) - \cos(x)\cos(y) + \sin(x)\sin(y)$$

By examining $f''(x)+f(x)$, use the Mean Value Theorem and the definitions of $\sin, \cos$ to prove:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \qquad \forall x, y \in \mathbb{R}.$$ (6 marks)

(ii) Use the previous result and the definitions of $\sin, \cos$ to prove:

$$\sin^2(x) + \cos^2(x) = 1 \qquad \forall x \in \mathbb{R}.$$ (4 marks)

(b) (i) Using a series expansion or otherwise, prove for any $m > 0$:

$$\lim_{x\to\infty} \frac{x^m}{e^x} = 0.$$ (5 marks)

(ii) Use the previous result and integration by parts to prove the following property of the Gamma function Γ:

$$\Gamma(x+1) = x\Gamma(x) \qquad \forall x \in (0, \infty)$$ (5 marks)

(iii) Use the previous result to prove by mathematical induction:

$$\Gamma(n+1) = n! \qquad \forall n \in \{0, 1, 2, \ldots\}.$$ (5 marks)

END OF PAPER

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Definitions and Theorems

- Limit of a function as x approaches $c \in \mathbb{R}$:

$$\lim_{x\to c} f(x) = L \quad \text{if for any } \epsilon > 0,\ \exists\delta > 0 \ :\ |x-a| < \delta \implies |f(x)-L| < \epsilon$$

- Limit of a function as x diverges to $+\infty$:

$$\lim_{x\to\infty} f(x) = L \quad \text{if for any } \epsilon > 0,\ \exists\delta > 0 \ :\ x > \delta \implies |f(x)-L| < \epsilon$$

- Continuity of a function at $x = c$:

$$\lim_{x\to c} f(x) = f(c)$$

- Lipschitz continuity of a function on $\mathcal{I} \subset \mathbb{R}$ with Lipschitz constant $L > 0$:

$$|f(x) - f(y)| \leq L|x-y| \quad \forall x, y \in \mathcal{I}$$

- Extreme Value Theorem:

  Let f be a continuous real function on $[a, b]$, then f attains its maximum and minimum on $[a, b]$.

- Intermediate Value Theorem:

  Let f be a continuous real function on $[a, b]$.

$$\gamma \in \mathbb{R} \ :\ f(a) < \gamma < f(b) \implies \exists c \in (a,b) \ :\ f(c) = \gamma.$$

- Derivative of f at $x = c$:

  Let f be a real function on $[a, b]$ and for fixed $x \in [a, b]$ define the quotient:

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (t \in (a,b),\ t \neq x)$$

  then

$$f'(x) = \lim_{t\to x} \phi(t).$$

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- Mean Value Theorem:

  Let f be a continuous real function on $[a, b]$ and differentiable on (a, b):

$$\exists c \in (a, b) \ : \ f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Taylor's Theorem:

  Let f be a real function on $[a, b]$ such that for fixed $n \in \mathbb{N}$, $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}(t)$ exists for all $t \in (a, b)$. Let $x, \alpha \in (a, b)$ be distinct, then there exists $\xi \in (a, b)$ with $\alpha < \xi < x$ such that:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x - \alpha)^k + \frac{f^{(n)}(\xi)}{n!}(x - \alpha)^n.$$

- Lower and upper Riemann sums:

  Given a partition $P_N[a, b] = \{a = x_0 < x_1 < \cdots < x_N = b\}$ we define:

$$\begin{aligned} m_i &= \inf\{f(x) \ : \ x \in [x_{i-1}, x_i]\} \\ M_i &= \sup\{f(x) \ : \ x \in [x_{i-1}, x_i]\} \\ \Delta x_i &= x_i - x_{i-1} \end{aligned}$$

  which yield the lower and upper Riemann sums:

$$\mathcal{L}(P_N, f) = \sum_{i=1}^{N} m_i \Delta x_i \quad \text{and} \quad \mathcal{U}(P_N, f) = \sum_{i=1}^{N} M_i \Delta x_i$$

  respectively

- Riemann's integrability condition:

  Let f be a bounded real function on $[a, b]$.

f is Riemann integrable on $[a, b]$ $\iff$ for any $\epsilon > 0$, $\exists P_N[a, b]$ such that $\mathcal{U}(P_N, f) - \mathcal{L}(P_N, f) < \epsilon$

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- Definitions of $\cos, \sin, \exp$ in terms of power series:

$$\cos(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad \sin(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad \exp(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

These power series are absolutely convergent and differentiable on $\mathbb{R}$.

- Gamma function:

The Gamma function Γ is defined by:

$$\Gamma(x) = \int_{t=0}^{\infty} t^{x-1} e^{-t} \, dt$$

and converges for all $x \in (0, \infty)$.