PAST EXAMINATION PAPER
[ESS02]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

BSc (Hons) MATHEMATICS

SEMESTER 2 EXAMINATIONS 2018/19

LINEAR ALGEBRA

MODULE NO: MMA5005

Date: Wednesday 22 May 2019 **Time: 10.00-12.15**

INSTRUCTIONS TO CANDIDATES:

1. Answer all FOUR questions.
2. All questions carry equal marks.
3. Maximum marks for each part/question are shown in brackets.

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1. (a) Show that the following vectors in $\boldsymbol{R}^5$ are linearly independent:

$$\begin{pmatrix}1\\0\\2\\0\\3\end{pmatrix}, \begin{pmatrix}0\\2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\\0\\4\end{pmatrix}$$

State, with reasons, whether or not these vectors form a basis for $\boldsymbol{R}^5$

(8 marks)

(b) A mapping $f: \boldsymbol{R}^3 \to \boldsymbol{R}^2$ is defined by

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+2y+z\\-x+y+3z\end{pmatrix}.$$

Show that f is a linear transformation.

(5 marks)

Suppose that the basis for the domain is changed to

$$\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$$

and the basis for the codomain is changed to $\left\{\begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$

Write down the transition matrices for these two bases.

Calculate the matrix that represents f with respect to the new bases.

(6 marks)

(c) Suppose that $f: V \to W$ is a linear transformation of real vector spaces.

State what is meant by the kernel, $\ker f$.

Prove that f is injective if and only if $\ker f = \{\mathbf{0}\}$.

(6 marks)

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2. (a) Find the eigenvalues of the following matrix:

$$M = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & -2 & 3 \end{pmatrix}$$

State the algebraic multiplicity of each eigenvalue.

(8 marks)

Find the eigenvectors of the matrix M. Explain why M cannot be diagonalised.

(8 marks)

(b) Suppose that $f : \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^2$ is a linear transformation with distinct eigenvalues λ_1 and λ_2 and corresponding eigenvectors v_1 and v_2.

Show that with respect to the basis $\{v_1, v_2\}$ the linear transformation f is represented by the diagonal matrix:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(9 marks)

3. (a) For vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \boldsymbol{R}^2$ define

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = 2x_1x_2 + 3y_1y_2.$$

Show that this is an inner product on $\boldsymbol{R}^2$.

State, with reasons, whether or not

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1y_1 + x_2y_2$$

defines an inner product on $\boldsymbol{R}^2$

(12 marks)

(b) Consider the following vectors in $\boldsymbol{R}^3$:

$$u_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, u_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Show that $\{u_1, u_2, u_3\}$ is a basis for $\boldsymbol{R}^3$.

(4 marks)

(c) Beginning with the basis of part (b) above, use the Gram-Schmidt process to obtain a basis $\{v_1, v_2, v_3\}$ which is orthogonal with respect to the standard inner product on $\boldsymbol{R}^3$.

(9 marks)

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4. (a) Let V be a real inner product space and let $S = \{v_1, v_2, ..., v_n\}$ be a set of non-zero vectors in V.

Explain what it means for the set S to be orthogonal.

Show that if S is orthogonal then S is linearly independent.

(8 marks)

(b) Consider the linear transformation $f: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9x + 5y \\ 5x - 15y \end{pmatrix}$$

where $\boldsymbol{R}^2$ is an inner product space under the standard scalar product.

Show that f is a symmetric linear transformation.

(6 marks)

Write down the matrix M which represents f with respect to the standard basis for $\boldsymbol{R}^2$.

Find the eigenvalues and eigenvectors of M.

(8 marks)

Hence find an orthonormal basis of $\boldsymbol{R}^2$ of eigenvectors.

Write down the transition matrix P for this basis.

Write down the matrix $P^T MP$, and explain what the entries on the leading diagonal represent.

(3 marks)

END OF QUESTIONS