[ENG09]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

Bsc (Hons) MATHEMATICS

SEMESTER 1 EXAMINATIONS 2018/19

ABSTRACT ALGEBRA

MODULE NO: MMA4001

Date: Tuesday 15 January 2019 **Time:10.00am - 12.15pm**

INSTRUCTIONS TO CANDIDATES:

1. **Answer all FOUR questions.**
2. **All questions carry equal marks.**
3. **Maximum marks for each part/question are shown in brackets.**

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1. (a) State what is meant by a binary operation $*$ on a set S. Describe the properties that the pair $(S, *)$ must have in order to be a *group*.

(8 marks)

(b) Consider the binary operation $*$ on the set $S = \{a, b, c\}$ defined by the following Cayley table:

$*$	a	b	c
a	a	b	c
b	b	c	b
c	c	b	a

State, with reasons, whether or not this operation is commutative and whether or not this operation is associative. State the identity, and state the inverse of each element where defined.

(7 marks)

(c) Find the centraliser of the matrix $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ in the general linear group $GL(2, \boldsymbol{Q})$.

(10 marks)

Please turn the page.

2. (a) Show that the set of matrices

$$H = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a, b \in \boldsymbol{Q},\ a \neq 0 \right\}$$

is a subgroup of $GL(2, \boldsymbol{Q})$.

State, with reasons, whether or not

$$H' = \left\{ \begin{pmatrix} 0 & a \\ a & b \end{pmatrix} : a, b \in \boldsymbol{Q},\ a \neq 0 \right\}$$

is a subgroup of $GL(2, \boldsymbol{Q})$.

(10 marks)

(b) Let $(\boldsymbol{R}, +)$ be the group of real numbers under addition and let $(\boldsymbol{R}^*, \cdot)$ be the group of non-zero real numbers under multiplication.

Show that the mapping $f : \boldsymbol{R} \to \boldsymbol{R}^*$ given by $f(x) = 3^x$ is a homomorphism of groups.

State the image $\operatorname{im} f$ of this homomorphism.

(5 marks)

(c) Let $f : G \to H$ be a homomorphism of groups.

State what is meant by the kernel, $\ker f$.

Show that if $\ker f = \{e_G\}$, where e_G is the identity of G, then f is injective.

(10 marks)

Please turn the page

3. (a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$ and $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 2 & 4 & 5 \end{pmatrix}$ be permutations in S_6.

Write down the following permutations

$\sigma \circ \pi \qquad \pi \circ \sigma \qquad \sigma^{-1} \qquad \pi^2.$

Find the permutation ρ such that

$$\sigma \circ \rho = \pi.$$

(10 marks)

(b) Express the following products of cycles as permutations on S_7:

$$\begin{aligned} \alpha &= (246)(135) \\ \beta &= (17)(26)(35) \end{aligned}$$

Hence, or otherwise, find $\alpha \circ \beta$ as a product of disjoint cycles.

(7 marks)

(c) Draw the Cayley table for the group $(\mathbf{Z}_{10}, +)$.

State the inverse of each element.

(8 marks)

Please turn the page

4. (a) Consider the ring $\boldsymbol{Q}[x]$ of polynomials with rational number coefficients.

State, with reasons, whether or not $\boldsymbol{Q}[x]$ is

(i) an integral domain

(ii) a field.

(6 marks)

(b) Using the formula $e^{i\theta} = \cos\theta + i\sin\theta$ prove the following trigonometric identities:

$$\cos(\theta + \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$
$$\sin(\theta + \psi) = \cos\theta\sin\psi + \sin\theta\cos\psi$$

(8 marks)

(c) Solve the quadratic equation

$$x^2 + 6x + 34 = 0.$$

Indicate the roots on an Argand diagram.

(5 marks)

(d) Let $z_1 = 5 + 12i$ and $z_2 = 9 - 2i$ be complex numbers.

Express each of these in polar form.

Hence find the following in polar form:

(i) $z_1 z_2$ (ii) $\dfrac{z_1}{z_2}$ (iii) $\sqrt{z_1}$

(6 marks)

END OF QUESTIONS